13-6 Circular Functions

Main Ideas

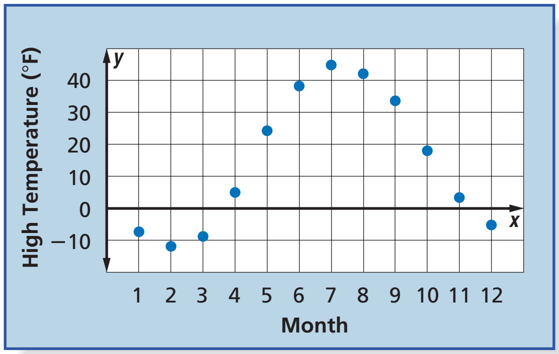
- Define and use the trigonometric functions based on the unit circle.
- Find the exact values of trigonometric functions of angles.

New Vocabulary

circular function
periodic
period

GET READY for the Lesson

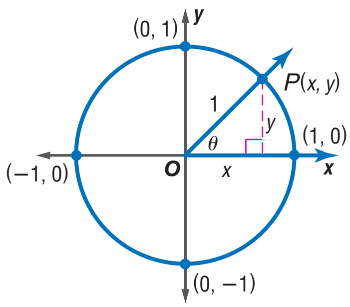
The average high temperatures, in degrees Fahrenheit, for Barrow, Alaska, are given in the table at the right. With January assigned a value of 1, February a value of 2, March a value of 3, and so on, these data can be graphed as shown below. This pattern of temperature fluctuations repeats after a period of 12 months.



MONTH	HIGH TEMP. (°F)
Jan	-7.4
Feb	-11.8
March	-9.0
April	4.7
May	24.2
June	38.3
July	45.0
Aug	42.3
Sept	33.8
Oct	18.1
Nov	3.5
Dec	-5.2

Source: www.met.utah.edu

Unit Circle Definitions From your work with reference angles, you know that the values of trigonometric functions also repeat. For example, sin 30° and sin 150° have the same value, $\frac{1}{2}$. In this lesson, we will further generalize the functions by defining them in terms of the unit circle.

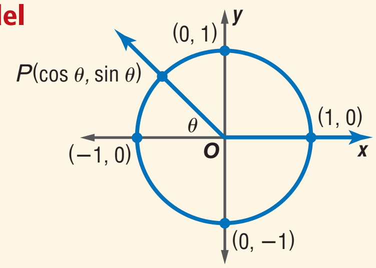


Consider an angle θ in standard position. The terminal side of the angle intersects the unit circle at a unique point, $P(x, y)$. Recall that $\sin \theta = \frac{y}{r}$ and $\cos \theta = \frac{x}{r}$. Since $P(x, y)$ is on the unit circle, $r = 1$. Therefore, $\sin \theta = y$ and $\cos \theta = x$.

KEY CONCEPT — *Definition of Sine and Cosine*

Words If the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$, then $\cos \theta = x$ and $\sin \theta = y$. Therefore, the coordinates of P can be written as $P(\cos \theta, \sin \theta)$.

Model

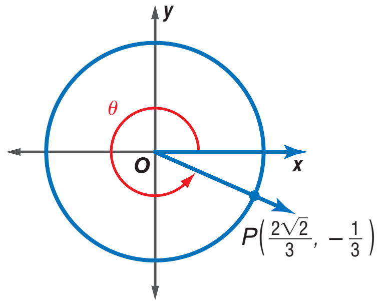


Since there is exactly one point $P(x, y)$ for any angle θ, the relations $\cos \theta = x$ and $\sin \theta = y$ are functions of θ. Because they are both defined using a unit circle, they are often called **circular functions**.

> **Study Tip**
>
> **Remembering Relationships**
>
> To help you remember that $x = \cos \theta$ and $y = \sin \theta$, notice that alphabetically x comes before y and cosine comes before sine.

EXAMPLE Find Sine and Cosine Given Point on Unit Circle

1 **Given an angle θ in standard position, if $P\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right)$ lies on the terminal side and on the unit circle, find $\sin \theta$ and $\cos \theta$.**



$P\left(\frac{2\sqrt{2}}{3}, -\frac{1}{3}\right) = P(\cos \theta, \sin \theta)$,

so $\sin \theta = -\frac{1}{3}$ and $\cos \theta = \frac{2\sqrt{2}}{3}$.

CHECK Your Progress

1. Given an angle θ in standard position, if $P\left(\frac{\sqrt{6}}{5}, \frac{\sqrt{19}}{5}\right)$ lies on the terminal side and on the unit circle, find $\sin \theta$ and $\cos \theta$.

GRAPHING CALCULATOR LAB

Sine and Cosine on the Unit Circle

Press MODE and highlight Degree and Par. Then use the following range values to set up a viewing window: TMIN = 0, TMAX = 360, TSTEP = 15, XMIN = −2.4, XMAX = 2.35, XSCL = 0.5, YMIN = −1.5, YMAX = 1.55, YSCL = 0.5.

Press Y= to define the unit circle with $X_{1T} = \cos T$ and $Y_{1T} = \sin T$.

Press GRAPH . Use the TRACE function to move around the circle.

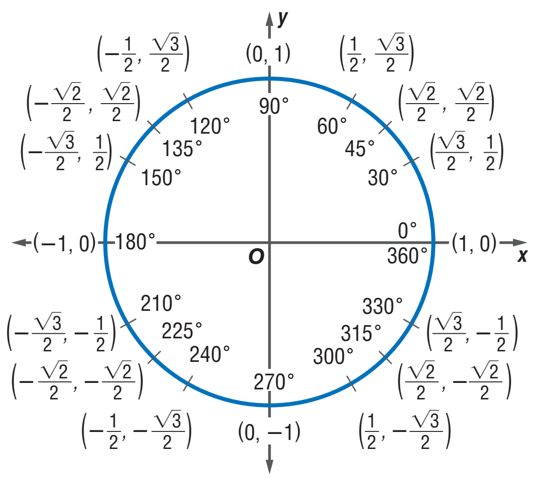
THINK AND DISCUSS

1. What does T represent? What do the x- and y-values represent?

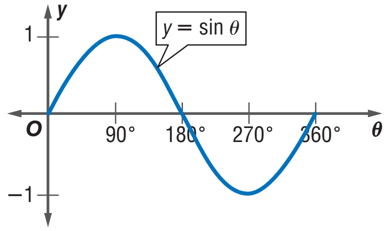
2. Determine the sine and cosine of the angles whose terminal sides lie at 0°, 90°, 180°, and 270°.

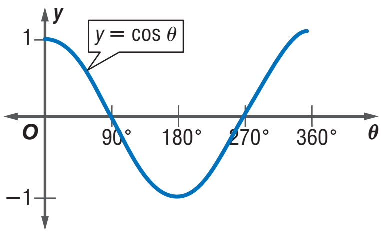
3. How do the values of sine change as you move around the unit circle? How do the values of cosine change?

The exact values of the sine and cosine functions for specific angles are summarized using the definition of sine and cosine on the unit circle at the right.

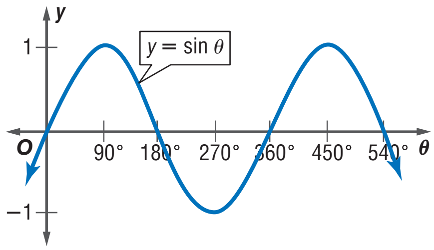


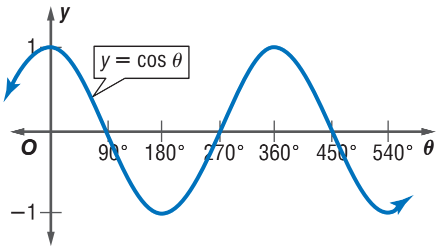
This same information is presented on the graphs of the sine and cosine functions below, where the horizontal axis shows the values of θ and the vertical axis shows the values of $\sin \theta$ or $\cos \theta$.





Periodic Functions Notice in the graph above that the values of sine for the coterminal angles 0° and 360° are both 0. The values of cosine for these angles are both 1. Every 360° or 2π radians, the sine and cosine functions repeat their values. So, we can say that the sine and cosine functions are **periodic**, each having a **period** of 360° or 2π radians.





KEY CONCEPT *Periodic Function*

A function is called periodic if there is a number a such that $f(x) = f(x + a)$ for all x in the domain of the function. The least positive value of a for which $f(x) = f(x + a)$ is called the period of the function.

For the sine and cosine functions, $\cos (x + 360°) = \cos x$, and $\sin (x + 360°) = \sin x$. In radian measure, $\cos (x + 2\pi) = \cos x$, and $\sin (x + 2\pi) = \sin x$. Therefore, the period of the sine and cosine functions is 360° or 2π.

EXAMPLE Find the Value of a Trigonometric Function

2 **Find the exact value of each function.**

a. $\cos 675°$

$$\cos 675° = \cos (315° + 360°)$$
$$= \cos 315°$$
$$= \frac{\sqrt{2}}{2}$$

b. $\sin \left(-\frac{5\pi}{6}\right)$

$$\sin \left(-\frac{5\pi}{6}\right) = \sin \left(-\frac{5\pi}{6} + 2\pi\right)$$
$$= \sin \frac{7\pi}{6}$$
$$= -\frac{1}{2}$$

CHECK Your Progress

2A. $\cos \left(-\frac{3\pi}{4}\right)$

2B. $\sin 420°$

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When you look at the graph of a periodic function, you will see a repeating pattern: a shape that repeats over and over as you move to the right on the x-axis. The period is the distance along the x-axis from the beginning of the pattern to the point at which it begins again.

Many real-world situations have characteristics that can be described with periodic functions.

Real-World EXAMPLE Find the Value of a Trigonometric Function

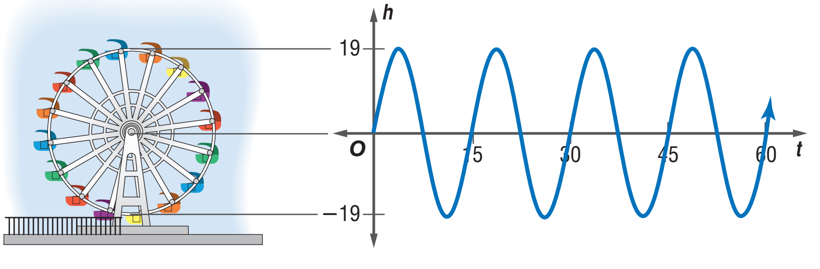
3 FERRIS WHEEL As you ride a Ferris wheel, the height that you are above the ground varies periodically as a function of time. Consider the height of the center of the wheel to be the starting point. A particular wheel has a diameter of 38 feet and travels at a rate of 4 revolutions per minute.

a. Identify the period of this function.

Since the wheel makes 4 complete counterclockwise rotations every minute, the period is the time it takes to complete one rotation, which is $\frac{1}{4}$ of a minute or 15 seconds.

b. Make a graph in which the horizontal axis represents the time *t* in seconds and the vertical axis represents the height *h* in feet in relation to the starting point.

Your height is 0 feet at the starting point. Since the diameter of the wheel is 38 feet, the wheel reaches a maximum height of $\frac{38}{2}$ or 19 feet above the starting point and a minimum of 19 feet below the starting point.



Because the period of the function is 15 seconds, the pattern of the graph repeats in intervals of 15 seconds on the x-axis.

Real-World Link

The Ferris Wheel was designed by bridge builder George W. Ferris in 1893. It was designed to be the landmark of the World's Fair in Chicago in 1893.

Source: National Academy of Sciences

CHECK Your Progress

A new model of the Ferris wheel travels at a rate of 5 revolutions per minute and has a diameter of 44 feet.

3A. What is the period of this function?

3B. Graph the function.

CHECK Your Understanding

Example 1 (p. 800)

If the given point P is located on the unit circle, find $\sin \theta$ and $\cos \theta$.

1. $P\left(\frac{5}{13}, -\frac{12}{13}\right)$ **2.** $P\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

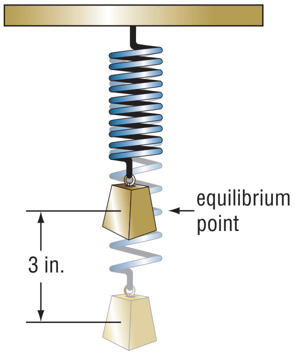
Example 2 (p. 801)

Find the exact value of each function.

3. $\sin -240°$ **4.** $\cos \frac{10\pi}{3}$

Example 3 (p. 802)

PHYSICS For Exercises 5 and 6, use the following information.
The motion of a weight on a spring varies periodically as a function of time. Suppose you pull the weight down 3 inches from its equilibrium point and then release it. It bounces above the equilibrium point and then returns below the equilibrium point in 2 seconds.



5. Find the period of this function.

6. Graph the height of the spring as a function of time.

Exercises

HOMEWORK HELP	
For Exercises	See Examples
7–12	1
13–18	2
19–38	3

The given point P is located on the unit circle. Find $\sin \theta$ and $\cos \theta$.

7. $P\left(-\frac{3}{5}, \frac{4}{5}\right)$ **8.** $P\left(-\frac{12}{13}, -\frac{5}{13}\right)$ **9.** $P\left(\frac{8}{17}, \frac{15}{17}\right)$

10. $P\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$ **11.** $P\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ **12.** $P(0.6, 0.8)$

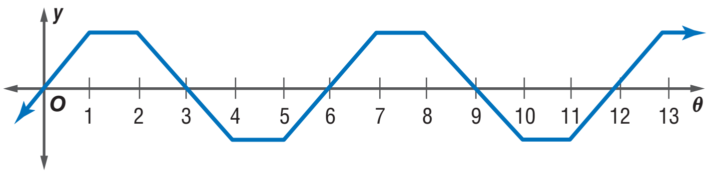
Find the exact value of each function.

13. $\sin 690°$ **14.** $\cos 750°$ **15.** $\cos 5\pi$

16. $\sin\left(\frac{14\pi}{6}\right)$ **17.** $\sin\left(-\frac{3\pi}{2}\right)$ **18.** $\cos(-225°)$

Determine the period of each function.

19.



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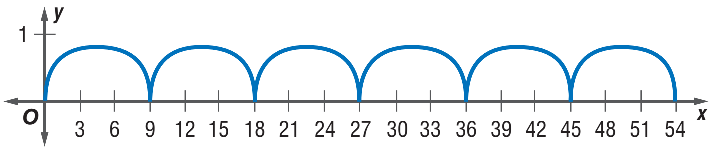

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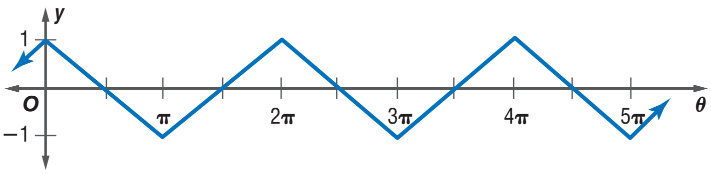
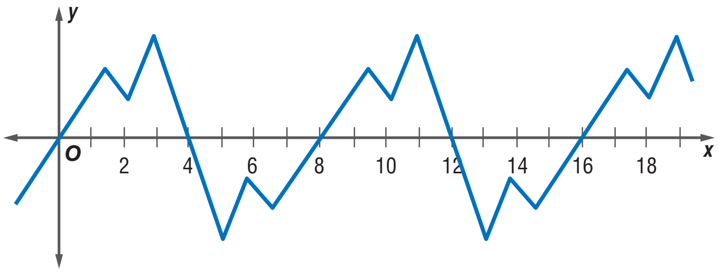


Determine the period of the function.

22.



Real-World Link

Most guitars have six strings. The frequency at which one of these strings vibrates is controlled by the length of the string, the amount of tension on the string, the weight of the string, and the springiness of the strings' material.

Source: www.howstuffworks.com

GUITAR **For Exercises 23 and 24, use the following information.**
When a guitar string is plucked, it is displaced from a fixed point in the middle of the string and vibrates back and forth, producing a musical tone. The exact tone depends on the frequency, or number of cycles per second, that the string vibrates. To produce an A, the frequency is 440 cycles per second, or 440 hertz.

23. Find the period of this function.

24. Graph the height of the fixed point on the string from its resting position as a function of time. Let the maximum distance above the resting position have a value of 1 unit and the minimum distance below this position have a value of 1 unit.

Find the exact value of each function.

25. $\dfrac{\cos 60° + \sin 30°}{4}$

26. $3(\sin 60°)(\cos 30°)$

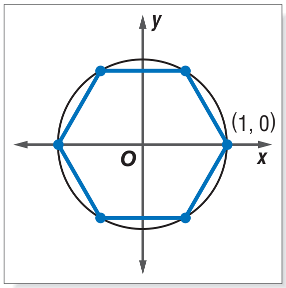
27. $\sin 30° - \sin 60°$

28. $\dfrac{4 \cos 330° + 2 \sin 60°}{3}$

29. $12(\sin 150°)(\cos 150°)$

30. $(\sin 30°)^2 + (\cos 30°)^2$

31. **GEOMETRY** A regular hexagon is inscribed in a unit circle centered at the origin. If one vertex of the hexagon is at (1, 0), find the exact coordinates of the remaining vertices.



32. **BIOLOGY** In a certain area of forested land, the population of rabbits R increases and decreases periodically throughout the year. If the population can be modeled by $R = 425 + 200 \sin\left[\frac{\pi}{365}(d - 60)\right]$, where d represents the dth day of the year, describe what happens to the population throughout the year.

SLOPE **For Exercises 33–38, use the following information.**
Suppose the terminal side of an angle θ in standard position intersects the unit circle at $P(x, y)$.

33. What is the slope of $\overline{OP}$?

34. Which of the six trigonometric functions is equal to the slope of $\overline{OP}$?

35. What is the slope of any line perpendicular to $\overline{OP}$?

36. Which of the six trigonometric functions is equal to the slope of any line perpendicular to $\overline{OP}$?

37. Find the slope of $\overline{OP}$ when $\theta = 60°$.

38. If $\theta = 60°$, find the slope of the line tangent to circle O at point P.

EXTRA PRACTICE
See pages 921, 938.
Math Online
Self-Check Quiz at algebra2.com

H.O.T. Problems

39. OPEN ENDED Give an example of a situation that could be described by a periodic function. Then state the period of the function.

40. WHICH ONE DOESN'T BELONG? Identify the expression that does not belong with the other three. Explain your reasoning.

$\sin 90°$	$\tan \frac{\pi}{4}$	$\cos 180°$	$\csc \frac{\pi}{2}$

41. CHALLENGE Determine the domain and range of the functions $y = \sin \theta$ and $y = \cos \theta$.

42. *Writing in Math* If the formula for the temperature T in degrees Fahrenheit of a city t months into the year is given by $T = 50 + 25 \sin\left(\frac{\pi}{6}t\right)$, explain how to find the average temperature and the maximum and minimum predicted over the year.

STANDARDIZED TEST PRACTICE

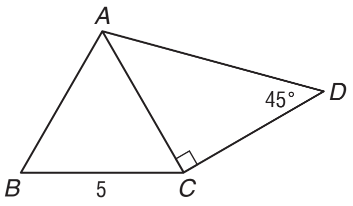
43. ACT/SAT If $\triangle ABC$ is an equilateral triangle, what is the length of $\overline{AD}$, in units?

A 5

B $5\sqrt{2}$

C 10

D $10\sqrt{2}$



44. REVIEW For which measure of θ is $\theta = \frac{\sqrt{3}}{3}$?

F 135°

G 270°

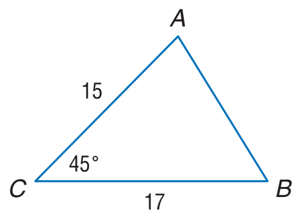
H 1080°

J 1830°

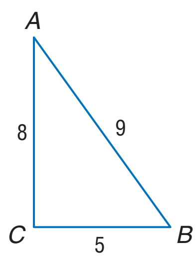
Spiral Review

Determine whether each triangle should be solved by beginning with the Law of Sines or Law of Cosines. Then solve each triangle. Round measures of sides to the nearest tenth and measures of angles to the nearest degree. (Lesson 13-5)

45.



46.



Find the area of $\triangle ABC$. Round to the nearest tenth. (Lesson 13-4)

47. $a = 11$ in., $c = 5$ in., $B = 79°$

48. $b = 4$ m, $c = 7$ m, $A = 63°$

48. BULBS The lifetimes of 10,000 light bulbs are normally distributed. The mean lifetime is 300 days, and the standard deviation is 40 days. How many light bulbs will last between 260 and 340 days? (Lesson 12-7)

Find the sum of each infinite geometric series, if it exists. (Lesson 11-5)

49. $a_1 = 3, r = 1.2$

50. $16, 4, 1, \frac{1}{4}, \ldots$

51. $\sum_{n=1}^{\infty} 13(-0.625)^{n-1}$

GET READY for the Next Lesson

PREREQUISITE SKILL Find each value of θ. Round to the nearest degree. (Lesson 13-1)

52. $\sin \theta = 0.3420$

53. $\cos \theta = -0.3420$

54. $\tan \theta = 3.2709$